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On the Relation between Global Properties of Linear Difference and Differential Equations with Polynomial Coefficients, I

G. K. Immink

Institute of Econometrics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

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This paper is concerned with applications of the Mellin transformation in the study of homogeneous linear differential and difference equations with polynomial coefficients. We begin by considering a differential equation (D) with regular singularities at O and ∞ and arbitrary singularities in the rest of the complex plane, and the difference equation (Δ') obtained from (D) by a variant of the formal Mellin transformation. We define fundamental systems of solutions of (Δ'), analytic in either a right or a left half plane, by the use of Mellin transforms of microsolutions of (D). The relations between these fundamental systems are expressed in terms of central connection matrices of (D). Second, we study the differential equation (D_1) obtained from (D) by means of a formal Laplace transformation and the difference equation (Δ_1) obtained from (D_1) by a formal Mellin transformation. We use Mellin transforms of "ordinary" solutions of (D_1) with moderate growth at ∞ to construct fundamental systems of solutions of (Δ_1). The relation between these fundamental systems involves certain Stokes multipliers and a formal monodromy matrix of (D_1).

0. Introduction

Linear difference equations and linear differential equations with polynomial coefficients are related through a formal Mellin transformation. Let τ denote the shift operator

$$\tau y(x) = y(x+1).$$

The formal Mellin transformation $\mathscr{M}: \mathbb{C}[u, u(d/du)] \to \mathbb{C}[x, \tau]$ is defined by the substitutions

$$u \to \tau, \qquad u\frac{d}{du} \to -x.$$

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It carries the differential equation

$$D\varphi := \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} u^h \left(u \frac{d}{du}\right)^l \varphi = 0 \tag{D}$$

into the difference equation

$$\Delta y(x) := \sum_{h=0}^{M} \sum_{l=0}^{m} (-1)^l a_{hl} (x+h)^l \, y(x+h) = 0. \tag{Δ}$$

Moreover, if $\sum_{n=-\infty}^{\infty} y(n) u^{-n}$ is a formal solution of (D) then the coefficients $y(n)$ satisfy the corresponding difference equation (Δ) and vice versa.

Alternatively, one may consider the transformation $\mathscr{P}$ defined by

$$u \to \tau^{-1}, \qquad u \frac{d}{du} \to x-1, \tag{0.1}$$

which changes (D) into

$$\Delta' y(x) := \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} (x-h-1)^l \, y(x-h) = 0. \tag{Δ'}$$

Obviously, $\Delta' y(x) = 0$ if and only if $z(x) = y(1-x)$ is a solution of (Δ).

The relationship between (D) and (Δ) or (Δ') can be exploited in several ways. In the first place, examination of the difference equation (Δ) or (Δ') may provide useful information concerning the coefficients of formal solutions of (D). This is of particular interest in the study of the asymptotic behaviour of solutions of (D) in the case that (D) has an irregular singularity at ∞. There are many recent developments in this field, notably the theory of resurgent functions of Ecalle, where formal solutions of differential equations play a central role (cf. [1–3, 7, 12, 13]).

In the second place, analytic solutions of (D) can often be represented by suitable integral transforms (analytic inverse Mellin transforms) of solutions of (Δ). A well-known example is the representation of solutions of the hypergeometric differential equation by means of Barnes integrals of solutions of the corresponding difference equation.

Conversely, analytic Mellin transforms of solutions of (D) may be used to represent solutions of (Δ). An analytic Mellin transform of a function φ is an integral of the type

$$\int_{\gamma} \varphi(u) \, u^{x-1} \, du, \tag{0.2}$$

where γ is a suitable path in the complex plane. The simplest example is provided by the differential equation

$$u\frac{d\varphi}{du}+u\varphi=0 \tag{D_1}$$

and the corresponding difference equation

$$y(x+1)-xy(x)=0. \tag{Δ_1}$$

The Mellin transform $\int_0^\infty e^{-u}u^{x-1}\,du$ of the solution e^{-u} of (D_1) is the integral representation of the gamma function, which obviously satisfies (Δ_1). This last application of the Mellin transform has been an important tool in the study of linear difference equations with polynomial coefficients since the end of the last century (cf. [6, 5, 8, 9, 18, 20, 21]).

The main purpose of this paper is to construct fundamental systems of solutions of (Δ) or (Δ') that are analytic in either a left or a right half-plane and to study the relation between these systems. (A fundamental system of solutions of a difference equation of order M is a set of M solutions, linearly independent over the periodic functions of period 1.) In particular, we are interested in the way this connection problem is related to lateral or central connection problems for the corresponding differential equation (D). The classical approach, to construct such fundamental systems of solutions by the use of suitable Mellin transforms of particular solutions of (D), has some severe limitations. For one thing, the procedure to be followed strongly depends on the precise nature of the singularities of (D), and in each case a rather long and intricate argument is needed in order to prove that it does eventually yield a fundamental system of solutions of the difference equation. As a matter of fact, the difficulties arising here are similar to those encountered in the situation of two differential equations related through a formal Laplace tranformation and can be partly overcome by using, instead of ordinary solutions, microsolutions of the differential equation (D) (cf. [17, 16, 15]). Essentially, a microfunction Φ at a point $\lambda \in \mathbb{C}$ is an equivalence class of functions with the same type of singularity at λ. More precisely, it is the equivalence class modulo regular functions at λ of a function f which is holomorphic in a small neighbourhood of λ on the Riemann surface of $\log(u-\lambda)$. A microsolution Φ of (D) is the equivalence class of a function f satisfying an inhomogeneous equation of the type $Df=g$, where g is regular at λ (and thus equivalent to 0).

A different approach was proposed by Ramis (cf. [5, 21]). It consists in replacing the original difference equation (Δ) by an equation (Δ_1) obtained by the substitution

$$y(x)\to z(x):=\Gamma(x)^k\,y(x), \tag{0.3}$$

where $k \in \mathbb{N}$. If k is chosen sufficiently large, the differential equation (D_1) corresponding to (Δ_1) through a formal inverse Mellin transformation, will have at most two singularities: a regular one at O and an irregular one at ∞. Let $\mathscr{A}_D^{<0}$ denote the sheaf, on the circle S^1, of solutions of (D_1) that decrease exponentially as the variable tends to ∞ in some sector. The Mellin transform, taken over a suitable half line, of such a solution, will be a solution of the difference equation (Δ_1), analytic in a right half-plane. Moreover, the dimension of the cohomology group $H^1(S^1, \mathscr{A}_D^{<0})$ is known to be equal to the order of (Δ_1). Using the Mellin transforms of a suitable basis of $H^1(S^1, \mathscr{A}_D^{<0})$, one can construct a fundamental system of solutions of (Δ_1) and hence of (Δ), analytic in a right half-plane (cf. [5, 21]).

In Part I of this paper, we concentrate on the case that (D) has at most regular singularities at O and ∞ and arbitrary singularities in the rest of the complex plane. Due to the relative simplicity of this case, the results can be stated in a more explicit form than in the general case, which will be discussed in Part II.

In Section 3, we define fundamental systems of solutions of (Δ') by means of Mellin transforms of microsolutions of (D). It turns out that the total number of linearly independent microsolutions of (D) at its singular points $\in \mathbb{C} - \{0\}$ is precisely equal to the order of (Δ'). We show that the relations between different fundamental systems can be expressed in terms of central connection matrices of (D).

In Section 4, we apply the method proposed by Ramis to this type of equation. It suffices, in this case, to take $k = 1$ in (0.3) and the differential equation (D_1) can be obtained from (D) by means of a formal Laplace transformation, i.e., the transformation $\mathscr{L}: \mathbb{C}[u, d/du] \to \mathbb{C}[t, d/dt]$ defined by the substitutions

$$u \to -\frac{d}{dt}, \qquad \frac{d}{du} \to t. \tag{0.4}$$

More precisely, the following diagram is found to commute:

$$\begin{tikzcd}
\mathbb{C}\left[u, u\frac{d}{du}\right] \arrow[r, "\mathscr{P}"] \arrow[d, "\mathscr{L}"] & \mathbb{C}[x, \tau^{-1}] \arrow[d, "\gamma"] \\
\mathbb{C}\left[t, \frac{d}{dt}\right] \arrow[r, "\mathscr{M}"] & \mathbb{C}[x, \tau^{-1}].
\end{tikzcd}$$

Here γ denotes the mapping: $\gamma(P) = \Gamma(x) P \Gamma(x)^{-1}$ for all $P \in \mathbb{C}[x, \tau^{-1}]$. The relations between the four equations (D), (D_1) $(= \mathscr{L}D)$, (Δ) $(= \mathscr{P}D)$, and (Δ_1) $(= \mathscr{M}D_1)$ have been studied by several authors (cf. [5, 18, 2, 3, 19]). The main purpose of this section is to show how a second fundamental system of solutions of (Δ_1), analytic in a left half-plane, can be constructed by the use of Mellin transforms of solutions of (D_1) with moderate growth at ∞. The relation between the different fundamental systems can be expressed in terms of the Stokes multipliers (of level 1) at ∞ of (D_1), which, in their turn, can be derived from the central connection matrices of (D), defined in Section 3.

1. Preliminaries

By $\mathbb{C}_\infty$ we shall denote the Riemann surface of the logarithm.

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. By $S(\alpha, \beta)$ and $S[\alpha, \beta]$ we shall denote the following sectors:

$$S(\alpha, \beta) = \{u \in \mathbb{C}_\infty : \alpha < \arg u < \beta\}, \qquad S[\alpha, \beta] = \{u \in \mathbb{C}_\infty : \alpha \leqslant \arg u \leqslant \beta\}.$$

Definition 1.1. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with the property that $\lim_{n\to\infty} \operatorname{Re} \rho_n = \infty (\lim_{n\to\infty} \operatorname{Re} \rho_n = -\infty)$ and let

$$\hat{\varphi}(u) = \sum_{n=0}^{\infty} \sum_{h=0}^{H} \varphi_{nh} u^{\rho_n} \log^h u, \tag{1.1}$$

where $H \in \mathbb{N}$, $\varphi_{nh} \in \mathbb{C}$ for all $n \in \mathbb{N}_0$ and $h \in \{0, ..., H\}$.

We say that a function φ is represented asymptotically by $\hat{\varphi}$ as $u \to 0$ $(u \to \infty)$, $\arg u = \alpha$, and write

$$\varphi(u) \sim \hat{\varphi}(u), \qquad u \to 0 \;\; (u \to \infty), \qquad \arg u = \alpha,$$

if, for all $N \in \mathbb{N}_0$, the following condition holds:

$$\begin{aligned} R_N(\varphi; u) &:= \varphi(u) - \sum_{n:\, \operatorname{Re}_{\rho n} \leqslant N} \sum_{h=0}^{H} \varphi_{nh} u^{\rho_n} \log^h u \\ &= o(u^N) \qquad \text{as} \quad u \to 0, \qquad \arg u = \alpha; \\ (R_N(\varphi; u) &:= \varphi(u) - \sum_{n:\, \operatorname{Re}_{\rho n} \geqslant -N} \sum_{h=0}^{H} \varphi_{nh} u^{\rho_n} \log^h u \\ &= o(u^{-N}) \qquad \text{as} \quad u \to \infty, \qquad \arg u = \alpha). \end{aligned}$$

If Ψ is an analytic function on $\mathbb{C}_\infty$, we say that φ is represented asymptotically by $\Psi\hat{\varphi}$, if $\Psi^{-1}\varphi$ is represented asymptotically by $\hat{\varphi}$.

φ has moderate growth as $u \to 0$ $(u \to \infty)$, $\arg u = \alpha$, if there exists a real number τ such that

$$\varphi(u) = O(u^\tau), \qquad \text{as} \quad u \to 0 \;\; (u \to \infty), \qquad \arg u = \alpha.$$

φ has subexponential growth as $u \to 0$ ($u \to \infty$), $\arg u = \alpha$, if, for every $\varepsilon > 0$,

$$\varphi(u) = O\left(\exp \frac{\varepsilon}{u}\right) \quad \text{as} \quad u \to 0 \ (\varphi(u) = O(\exp \varepsilon u) \text{ as } u \to \infty),$$

$$\arg u = \alpha.$$

We shall say that any of the asymptotic properties defined above holds as $u \to 0$ ($u \to \infty$) in $S[\alpha, \beta]$ if it holds uniformly on

$$\{u \in S[\alpha, \beta] : |u| < R\} \qquad (\{u \in S[\alpha, \beta] : |u| > R\})$$

for some $R > 0$. We shall say that the asymptotic property holds as $u \to 0$ ($u \to \infty$) in $S(\alpha, \beta)$ if it holds as $u \to 0$ ($u \to \infty$) in $S[\alpha', \beta']$ for all $\alpha', \beta' \in \mathbb{R}$ such that $\alpha < \alpha' < \beta' < \beta$.

DEFINITION 1.2 (cf. also Definition 1.5). Let γ be a path in $\mathbb{C}_\infty$ and φ a continuous function on γ. The Mellin transform $\mathcal{M}_\gamma(\varphi)$ and the Laplace transform $\mathcal{L}_\gamma(\varphi)$ of φ are defined by

$$\mathcal{M}_\gamma(\varphi)(x) = \int_\gamma \varphi(u)\, u^{x-1}\, du, \qquad \mathcal{L}_\gamma(\varphi)(x) = \int_\gamma \varphi(u)\, e^{-ux}\, du$$

for all $x \in \mathbb{C}$ for which the integral exists.

Let $\lambda \in \mathbb{C}_\infty \cup \{0\}$, $\alpha \in \mathbb{R}$. By $\gamma_{\lambda,\alpha}$ we denote the half line from λ to ∞ in the direction α.

Propositions 1.3 and 1.4 below are easily deduced from the results stated in [4].

If (D) is a differential equation with regular singularities at O and ∞, its solutions may be represented by convergent series of the type (1.1) (both with $\lim_{n\to\infty} \operatorname{Re} \rho_n = \infty$ and with $\lim_{n\to\infty} \operatorname{Re} \rho_n = -\infty$). The Mellin transforms of such solutions are meromorphic functions, due to the following proposition.

PROPOSITION 1.3. *Let $\lambda \in \mathbb{C}_\infty$, $\arg \lambda = \alpha$ and let φ be a continuous function on the segment $(0, \lambda)$ (the half line $\gamma_{\lambda,\alpha}$), admitting an asymptotic representation of the form* (1.1), *where $\lim_{n\to\infty} \operatorname{Re} \rho_n = \infty$ ($\lim_{n\to\infty} \operatorname{Re} \rho_n = -\infty$) as $u \to 0$ ($u \to \infty$), $\arg u = \alpha$. Then $\mathcal{M}_{(0,\lambda)}(\varphi)$ ($\mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)$) can be continued analytically to a meromorphic function in $\mathbb{C}$, having poles at the points $-\rho_n$, $n \in \mathbb{N} \cup \{0\}$, with principal part*

$$\sum_{h=0}^{H} \varphi_{nh} h!\, (-1)^h\, (x + \rho_n)^{-h-1}.$$

This meromorphic function will also be denoted by $\mathcal{M}_{(0,\lambda)}(\varphi)$ ($\mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)$).

If at O φ is represented by a convergent series of the type (1.1), then its Mellin transform $\mathcal{M}_{(0,\lambda)}(\varphi)$ ($\mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)$) can be continued analytically to both an upper and a lower half plane by means of continuous deformation of the path of integration. More generally, we have:

PROPOSITION 1.4. *Let $\lambda \in \mathbb{C}_\infty$ and let φ be a continuous function on $(0, \lambda)$. Moreover, assume that φ is analytic in*

$$\tilde{D}(0, r) := \{u \in \mathbb{C}_\infty,\ |u| \leqslant r\},$$

where $0 < r \leqslant |\lambda|$, and φ satisfies an inequality of the form

$$|\varphi(u)| \leqslant Ce^{K|\arg u|}\,|u|^\tau, \qquad u \in \tilde{D}(0, r),$$

where $\tau \in \mathbb{R}$, C and K are positive numbers. Then the function ζ defined by

$$\zeta(x) = \mathcal{M}_{(0,\lambda)}(\varphi)(x)$$

is analytic in the half plane $\operatorname{Re} x > -\tau$ and can be continued analytically to the half planes $|\operatorname{Im} x| > K$. Let $\arg \lambda = \alpha$, $re^{i\alpha} = a$ and let $\Gamma_a^\pm$ denote the path

$$\Gamma_a^\pm = \{ae^{i\theta} : \theta \in (0, \pm\infty)\}. \tag{1.2}$$

The analytic continuation of ζ to the half plane $\pm\operatorname{Im} x > K$ may be represented by

$$\zeta(x) = \int_{\Gamma_a^\pm} \varphi(u)\, u^{x-1}\, du + \int_a^\lambda \varphi(u)\, u^{x-1}\, du.$$

For every $K' > K$ and $\tau' < \tau$ there exists a positive number C' such that

$$|\zeta(x)| \leqslant C'\,|\lambda^x|, \qquad \operatorname{Re} x \geqslant -\tau',$$
$$|\zeta(x)| \leqslant C'\,|a^x|, \qquad \pm\operatorname{Im} x \geqslant K'.$$

Let U be the contour consisting of the half line from $-iK' - \infty$ to $\tau' - iK'$, the segment $[\tau' - iK', \tau' + iK']$ and the half line from $\tau' + iK'$ to $iK' - \infty$. For all $u \in \tilde{D}(0, r)$ we have

$$\varphi(u) = \frac{1}{2\pi i}\int_U \zeta(x)\, u^{-x}\, dx. \tag{1.3}$$

DEFINITION 1.5. Let $\alpha \in \mathbb{R}$ and let φ be a continuous function on $\gamma_{0,\alpha}$ with moderate growth, both as $u \to 0$, $\arg u = \alpha$ and as $u \to \infty$, $\arg u = \alpha$. Suppose that there exists a $\lambda \in \gamma_{0,\alpha}$ and a domain S of $\mathbb{C}$ such that both $\mathcal{M}_{(0,\lambda)}(\varphi)$ and $\mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)$ are analytic in S or can be continued analytically to S. We shall denote the analytic continuations of $\mathcal{M}_{(0,\lambda)}(\varphi)$ and $\mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)$ by the same symbols and define

$$\mathcal{M}_{\gamma_{0,\alpha}}(\varphi)(x) = \mathcal{M}_{(0,\lambda)}(\varphi)(x) + \mathcal{M}_{\gamma_{\lambda,\alpha}}(\varphi)(x), \qquad x \in S.$$

Obviously, the above definition is independent of $\lambda \in \gamma_{0,\alpha}$.

2. Mellin and Laplace Transforms of Microfunctions

We begin by recalling the definition of a microfunction as it is given by Malgrange (cf. [14, 17, 15]).

Definition 2.1. Let $\lambda \in \mathbb{C}_\infty \cup \{0\}$, $r > 0$. We use the following notation. $\mathcal{O}(\lambda, r)$ is the ring of holomorphic functions in the disk $D(\lambda, r)$ with center λ and radius r. $\tilde{\mathcal{O}}(\lambda, r)$ is the ring of holomorphic functions on the universal covering $\tilde{D}(\lambda, r)$ of $D(\lambda, r) - \{\lambda\}$ with respect to some fixed base point. $T_{(\lambda)}$ will denote the action of the monodromy on $\tilde{\mathcal{O}}(\lambda, r)$ (i.e., the mapping obtained by analytic continuation of the elements of $\tilde{\mathcal{O}}(\lambda, r)$ on a positive loop about λ). We put

$$\mathscr{C}(\lambda, r) = \tilde{\mathcal{O}}(\lambda, r)/\mathcal{O}(\lambda, r)$$

and denote by $\operatorname{can}_{(\lambda)}$ the projection from $\tilde{\mathcal{O}}(\lambda, r)$ onto $\mathscr{C}(\lambda, r)$. Furthermore,

$$\operatorname{var}_{(\lambda)} : \mathscr{C}(\lambda, r) \to \tilde{\mathcal{O}}(\lambda, r)$$

is the unique mapping with the property that

$$\operatorname{var}_{(\lambda)} \circ \operatorname{can}_{(\lambda)} = T_{(\lambda)} - I.$$

The set $\mathscr{C}(\lambda)$ of microfunctions at λ is defined as the inductive limit

$$\mathscr{C}(\lambda) = \lim_{r \to 0} \mathscr{C}(\lambda, r).$$

Furthermore, we write

$$\mathcal{O}(\lambda) = \lim_{r \to 0} \mathcal{O}(\lambda, r)$$

and

$$\tilde{\mathcal{O}}(\lambda) = \lim_{r \to 0} \tilde{\mathcal{O}}(\lambda, r).$$

Definition 2.2. Let $\lambda \in \mathbb{C}_\infty$, $\Phi \in \mathscr{C}(\lambda)$, and $\alpha \in \mathbb{R}$. Suppose that $\operatorname{var}_{(\lambda)} \Phi$ can be continued analytically to a function φ, holomorphic on $\gamma_{\lambda,\alpha}$, and growing at most exponentially as $u \to \infty$ on $\gamma_{\lambda,\alpha}$. Let $\tilde{\varphi} \in \tilde{\mathcal{O}}(\lambda, r)$ such that $\mathrm{can}_{(\lambda)} \tilde{\varphi} = \Phi$. Let $\lambda' \in \gamma_{\lambda,\alpha} \cap \tilde{D}(\lambda, r)$ and let $C_{\lambda,\lambda'}$ be a small positive loop about λ, starting at λ'. The Laplace transform $\mathcal{L}_{\gamma_{\lambda,\alpha}}(\Phi)$ of Φ is defined by

$$\mathcal{L}_{\gamma_{\lambda,\alpha}}(\Phi)(t) = \int_{C_{\lambda,\lambda'}} \tilde{\varphi}(u)\, e^{-ut}\, du + \int_{\gamma_{\lambda',\alpha}} \varphi(u)\, e^{-ut}\, du.$$

It is easily verified that $\mathcal{L}_{\gamma_{\lambda,\alpha}}(\Phi)$ does not depend on the choice of λ' nor on that of $\tilde{\varphi}$. It is a holomorphic function in a half plane of the form $\mathrm{Re}(te^{i\alpha}) > C$, where $C \geqslant 0$.

In a similar manner, one can define Mellin transforms of microfunctions.

DEFINITION 2.3. With the notation of Definition 2.2, assume that $\mathrm{var}_{(\lambda)} \Phi$ can be continued analytically to a function φ with moderate growth as $u \to \infty$ on $\gamma_{\lambda,\alpha}$. Then the Mellin transform $\mathcal{M}_{\gamma_{\lambda,\alpha}}(\Phi)$ of Φ is defined by

$$\mathcal{M}_{\gamma_{\lambda,\alpha}}(\Phi)(x) = \int_{C_{\lambda,\lambda'}} \tilde{\varphi}(u)\, u^{x-1}\, du + \int_{\gamma_{\lambda',\alpha}} \varphi(u)\, u^{x-1}\, du,$$

and we put

$$\mathcal{M}_{\gamma_{\lambda,\alpha}}(\Phi)(1-x) = \mathcal{P}_{\gamma_{\lambda,\alpha}}(\Phi)(x)$$

Similarly, let γ be a path from λ to O admitting a limiting direction at O. Let $\lambda' \in \gamma \cap \tilde{D}(\lambda, r)$ and let γ' denote the path from λ' to O such that $\gamma' \subset \gamma$. If $\mathrm{var}_{(\lambda)} \Phi$ can be continued analytically to a function φ with moderate growth as $u \to 0$ on γ, the Mellin transform $\mathcal{M}_{\gamma}(\Phi)$ of Φ is defined by

$$\mathcal{M}_{\gamma}(\Phi)(x) = \int_{C_{\lambda,\lambda'}} \tilde{\phi}(u)\, u^{x-1}\, du + \int_{\gamma'} \varphi(u)\, u^{x-1}\, du,$$

and we put

$$\mathcal{M}_{\gamma}(\Phi)(1-x) = \mathcal{P}_{\gamma}(\Phi)(x)$$

Remark 2.4. Let $\Gamma_{\lambda,\alpha}$ denote the contour consisting of the half line $\gamma_{\lambda',\alpha}$ described in the opposite direction (i.e., from ∞ to λ'), the positive loop $C_{\lambda,\lambda'}$ from λ' to λ'', and the half line $\gamma_{\lambda'',\alpha+2\pi}$ described from λ'' to ∞. Suppose that $\tilde{\varphi}$ can be continued analytically to a holomorphic function on $\Gamma_{\lambda,\alpha}$ with at most exponential growth of order 1 as $u \to \infty$ on $\Gamma_{\lambda,\alpha}$. Then we have

$$\mathcal{L}_{\gamma_{\lambda,\alpha}}(\Phi) = \mathcal{L}_{\Gamma_{\lambda,\alpha}}(\tilde{\varphi}). \tag{2.1}$$

Similarly, if $\tilde{\varphi}$ has moderate growth as $u \to \infty$ on $\Gamma_{\lambda,\alpha}$, we have

$$\mathcal{M}_{\gamma_{\lambda,\alpha}}(\Phi) = \mathcal{M}_{\Gamma_{\lambda,\alpha}}(\tilde{\varphi}). \tag{2.2}$$

Now suppose that (D) is a differential equation with polynomial coefficients having a regular singularity at ∞ and an arbitrary singularity at some point $\lambda \in \mathbb{C}_\infty$. Then, for suitable values of α, the mapping $\mathcal{M}_{\gamma_{\lambda,\alpha}}$ is a bijection from the space of microsolutions of (D) at λ onto the space of solutions of $(\mathcal{M}D)$ that are analytic in a left half-plane and satisfy a certain growth condition. Below we state the equivalent result for $(\mathcal{P}D)$ instead of $(\mathcal{M}D)$, where $\mathcal{P}$ is the mapping defined in (0.1).

PROPOSITION 2.5. *Let D be a linear differential operator of order m, with polynomial coefficients, having at most a regular singularity at ∞ and let $\Delta' = \mathcal{P}D$. Suppose that $\lambda \in \mathbb{C}_\infty$ is a singular point of D. Let $\alpha < \arg\lambda$ $(\alpha > \arg\lambda)$ such that the region $\alpha \leqslant \arg(u-\lambda) < \arg\lambda$ $(\arg\lambda < \arg(u-\lambda) \leqslant \alpha)$ does not contain any singular points of D.*

The mapping $\mathcal{P}_{\gamma_{\lambda,\alpha}}$ defined in Definition 2.3 is a bijection from $\operatorname{Ker}(D, \mathscr{C}(\lambda))$ onto the space $S_\lambda(\Delta')$ of meromorphic solutions y of (Δ') with the following properties:

(i) *The poles of y belong to the set*

$$P = \{\rho_q - n : q \in \{1, ..., m\}, n \in \mathbb{N}\},$$

where $\rho_1, ..., \rho_m$ are the characteristic exponents relative to the regular singularity of D at ∞.

(ii) *$\lambda^x y(x)$ has subexponential growth as $x \to \infty$ in $S[-\pi/2, \pi/2+\varepsilon]$ $(S[-\pi/2-\varepsilon, \pi/2])$ for some positive number ε.*

The inverse mapping can be defined as follows. Let $y \in S_\lambda(\Delta')$. Let $a > \operatorname{Re}\rho$ for all $\rho \in P$ and let

$$\tilde{\varphi}(u) = \frac{1}{2\pi i}\int_a^\infty y(x)\, u^{x-1}\, dx, \qquad |u| < |\lambda|,$$

$$\frac{\pi}{2} < \arg(u-\lambda) < \frac{3\pi}{2}. \tag{2.3}$$

$\tilde{\varphi}$ can be continued analytically to an element of $\tilde{\mathcal{O}}(\lambda, r)$ (which we denote again by $\tilde{\varphi}$) and $\mathcal{P}_{\gamma_{\lambda,\alpha}}^{-1}(y) = \operatorname{can}_{(\lambda)} \tilde{\varphi}$. Furthermore, let $\varphi_{(\alpha)}$ denote the branch of $\operatorname{var}_{(\lambda)} \mathcal{P}_{\gamma_{\lambda,\alpha}}^{-1}(y)$ obtained by analytic continuation along $\gamma_{\lambda,\alpha}$. Let R be a

positive number such that $D(0, R)$ contains all finite singular points of D. Then we have

$$\varphi_{(\alpha)}(u) = \frac{1}{2\pi i} \int_U y(x)\, u^{x-1}\, dx, \qquad |u| > R,$$

where U is a contour of the type used in (1.3) *enclosing the set P.*

Proof. By the substitution $u = \lambda v$ the theorem can be reduced to the case that $\lambda = 1$. We shall prove the theorem for $\alpha < 0$.

Let $\tilde{\varphi} \in \tilde{\mathcal{O}}(1, r)$ such that $\operatorname{can}_{(1)} \tilde{\varphi} = \Phi$ and let $\varphi = \operatorname{var}_{(1)} \Phi$. Obviously, $D\varphi = 0$. Therefore, φ may be continued analytically on any path that avoids the singular points of D. Let $\delta \in (0, r)$ and let $\lambda \in \gamma_{1,\alpha} \cap \tilde{D}(1, \delta)$. We put $y = y_1 + y_2$, where

$$y_1(x) = \int_{C_{1,\lambda}} \tilde{\varphi}(u)\, u^{-x}\, du, \qquad y_2(x) = \int_{\gamma_{\lambda,\alpha}} \varphi(u)\, u^{-x}\, du. \tag{2.4}$$

By means of partial integration, it is easily verified that y is a solution of (Δ'). Obviously, y_1 is an entire function of x and there exist positive numbers c (independent of δ) and C_δ such that

$$|y_1(x)| \leqslant C_\delta e^{c\delta|x|}, \qquad x \in \mathbb{C}. \tag{2.5}$$

According to Proposition 1.3, y_2 is a meromorphic function with poles at the points $\in P$.

For the sake of simplicity we assume that there are no singular points of (D) on the half line $\gamma_{1,0}$. Let ε be a positive number such that $\varphi(e^t)$ is holomorphic on the sector $-\varepsilon \leqslant \arg t \leqslant 0$. Due to the regular singularity of (D) at ∞, $\varphi(e^t)$ grows at most exponentially as $t \to \infty$ in this sector. It is easily seen that, for every $\beta \in [-\varepsilon, 0]$,

$$y_2(x) = \int_{\gamma_{\log \lambda, \beta}} \varphi(e^t)\, e^{-t(x-1)}\, dt, \qquad \operatorname{Re}(xe^{i\beta}) \geqslant C,$$

where C is a positive number. Moreover, y_2 is bounded on the closed sector

$$S := \bigcup_{\beta \in [-\varepsilon, 0]} \{x \in \mathbb{C} : \operatorname{Re}(xe^{i\beta}) \geqslant C\}.$$

(If the above assumption is not fulfilled, the half line $\gamma_{\log \lambda, 0}$ has to be replaced by a path that avoids the singular points of $\varphi(e^t)$. In that case y_2 has at most subexponential growth as $x \to \infty$ in S.) With (2.5), it follows that y has subexponential growth as $x \to \infty$ in S. Due to the fact that y is a solution of the difference equation (Δ'), the same is true as $x \to \infty$ in $S - n$ for every $n \in \mathbb{N}$. Consequently, y has subexponential growth as $x \to \infty$ in $S[-\pi/2, \pi/2 + \varepsilon]$.

Now suppose that $y \in S_1(\Delta')$. The function $\tilde{\varphi}$ defined by (2.3) is analytic in the region $\pi/2 < \arg(u-1) < 3\pi/2$, $|u| < 1$. It can be continued analytically by changing the path of integration. Thus, for example, we have

$$\tilde{\varphi}(u)=\frac{1}{2\pi i}\int_a^{a-i\infty} y(x)\,u^{x-1}\,dx, \qquad \arg u<0, \qquad \pi<\arg(u-1)<2\pi$$

$$\tilde{\varphi}(u)=\frac{1}{2\pi i}\int_a^{a+i\infty} y(x)\,u^{x-1}\,dx, \qquad \arg u>0, \qquad 0<\arg(u-1)<\pi.$$

More generally, the integral $\int_{\gamma_{a,\theta}} y(x)\,u^{x-1}\,dx$ converges for all u with the property that $\ln|u|\cos\theta-\arg u\sin\theta<0$. Let

$$V:=\left\{u\colon |\arg u|<\frac{\pi}{2},\ -\frac{\pi}{2}<\arg(u-1)<\pi,\ \ln|u|>\operatorname{tg}\left(\frac{\pi}{2}+\varepsilon\right)\arg u\right\}$$

$\tilde{\varphi}$ can be continued analytically into V by varying θ from $\pi/2$ to $\pi/2+\varepsilon$. Let $u\in V$, $\arg u<0$. By u', we denote the endpoint of a positive loop about 1, starting at u. We put

$$\tilde{\varphi}(u')-\tilde{\varphi}(u)=\varphi(u), \qquad u\in V, \qquad \arg u<0. \tag{2.6}$$

By means of partial integration it is easily verified that $D\tilde{\varphi}$ is regular at 1. Consequently, $\tilde{\varphi}$ can be continued to an element of $\tilde{\mathcal{O}}(1,r)$, provided r is sufficiently small, and thus defines a microsolution Φ of (D) at 1. Moreover, $\operatorname{var}_{(1)}\Phi=\varphi$.

Let $\lambda\in V\cap D(1,r)$ such that $\beta:=\arg\lambda\in[\alpha,0)$. We choose a point $u_1\in V\cap C_{1,\lambda}$ such that $|u_1|<1$ and a point $u_2\in C_{1,\lambda}$ such that $|u_2|<1$ and $\operatorname{Im}u_2<0$. Let C_1, C_2 and C_3 denote the arcs of $C_{1,\lambda}$ between λ and u_1, between u_1 and u_2, and between u_2 and λ' (the endpoint of $C_{1,\lambda}$), respectively. We choose $\theta\in(\pi/2,\pi/2+\varepsilon)$ such that $\ln|u|>\operatorname{tg}\theta\arg u$ for all $u\in C_1$. Then we have

$$\int_{C_1} du\,u^{-x}\tilde{\varphi}(u)=\frac{1}{2\pi i}\int_{C_1} du\,u^{-x}\int_{\gamma_{a,\theta}} d\xi\,u^{\xi-1}y(\xi).$$

If $\arg(x-a)\in(-\pi/2,\theta)$, the order of integration may be reversed and we obtain

$$\int_{C_1} du\,u^{-x}\tilde{\varphi}(u)=\frac{1}{2\pi i}\int_{\gamma_{a,\theta}} d\xi\,\frac{u_1^{\xi-x}-\lambda^{\xi-x}}{\xi-x}\,y(\xi).$$

Similarly, we have

$$\int_{C_2} du\,u^{-x}\tilde{\varphi}(u)=\frac{1}{2\pi i}\int_a^{\infty} d\xi\,\frac{u_2^{\xi-x}-u_1^{\xi-x}}{\xi-x}\,y(\xi)$$

provided $\operatorname{Im} x \neq 0$, and

$$\int_{C_3} du\, u^{-x}\tilde{\varphi}(u) = \frac{1}{2\pi i}\int_a^{a-i\infty} d\xi\, \frac{\lambda^{\xi-x} - u_2^{\xi-x}}{\xi - x}\, y(\xi).$$

Let C_θ denote the contour consisting of the half lines $(a-i\infty, a)$ and $\gamma_{a,\theta}$. Combining the above identities and applying Cauchy's theorem, we find

$$\int_{C_{1,\lambda}} du\, u^{-x}\tilde{\varphi}(u) = y(x) - \frac{1}{2\pi i}\int_{C_\theta} d\xi\, \frac{\lambda^{\xi-x}}{\xi - x}\, y(\xi) \tag{2.7}$$

for all x such that $\arg(x-a) \in (-\pi/2, 0) \cup (0, \theta)$. From (2.6), we deduce that

$$\varphi(u) = \frac{1}{2\pi i}\int_{C_\theta} d\xi\, u^{\xi-1} y(\xi), \qquad u \in \gamma_{\lambda,\beta}.$$

Hence it follows, by a change of the order of integration, that

$$\int_{\gamma_{\lambda,\beta}} du\, u^{-x}\varphi(u) = \frac{1}{2\pi i}\int_{C_\theta} d\xi\, \frac{\lambda^{\xi-x}}{\xi - x}\, y(\xi), \tag{2.8}$$

provided $\arg(x-a) \in (-\pi/2, \theta)$. From (2.7) and (2.8), we conclude that $y(x) = \mathscr{P}_{\gamma_{1,\beta}}(\Phi)(x)$. Obviously, $\mathscr{P}_{\gamma_{1,\alpha}}(\Phi) = \mathscr{P}_{\gamma_{1,\beta}}(\Phi)$. This shows that the mapping $\mathscr{P}_{\gamma_{1,\alpha}}$ is surjective. The injectivity can be proved in a similar manner.

Now, let y_1 and y_2 be defined by (2.4) and let U be a contour of the type used in (1.3), enclosing the set P. Again y_1 is an entire function, satisfying an inequality of the form (2.5). Let R be a positive number such that all singular points of D are contained in $D(0, R)$. Obviously,

$$\int_U y_1(x)\, u^{x-1}\, dx = 0, \qquad |u| > R,$$

provided δ is chosen sufficiently small. From Proposition 1.4, we deduce, by a change of variable $u \to 1/u$, that

$$\varphi_{(\alpha)}(u) = \frac{1}{2\pi i}\int_U y_2(x)\, u^{x-1}\, dx, \qquad |u| > R.$$

The last statement of the proposition now follows immediately. ∎

Remark 2.6. By the change of variable $u \to t := -\log u$, $\mathbb{C}_\infty$ is mapped conformally onto $\mathbb{C}$ and $\mathscr{C}(\lambda)$ is mapped one to one onto $\mathscr{C}(-\log\lambda)$. Furthermore, the Mellin transform $\mathscr{M}_{\gamma_{\lambda,\alpha}}(\Phi)$ of a microfunction $\Phi \in \mathscr{C}(\lambda)$ is transformed into a Laplace transform of the corresponding element $\Psi \in \mathscr{C}(-\log \lambda)$, provided other paths of integration than half lines are allowed in Definition 2.2. In particular, if $\alpha = \arg \lambda$, $\mathscr{M}_{\gamma_{\lambda,\alpha}}(\Phi) = \mathscr{L}_{\gamma_{-\log \lambda,0}}(\Psi)$.

For many purposes, it is more convenient to represent solutions of (Δ') by means of Laplace integrals than by Mellin transforms. Thus, for example, for almost all directions α, the Laplace transform $\mathscr{L}_{\gamma_{-\log \lambda,\alpha}}(\Psi)(1-x)$ represents a solution y with the property that $\lambda^x y(x)$ has subexponential growth as $x \to \infty$ in the sector $S[-\pi/2-\alpha-\varepsilon, \pi/2-\alpha+\varepsilon]$, where ε is some sufficiently small positive number (cf. also [11]). The corresponding path on $\mathbb{C}_\infty$ has a more complicated shape; it is a kind of spiral with continuously growing or decreasing radius.

3. The Equations (D) and (Δ')

We consider linear differential operators D of the form

$$D = \sum_{l=0}^{m} \sum_{h=0}^{M} a_{hl} u^h \left(u \frac{d}{du}\right)^l.$$

We shall assume that $a_{0m} \neq 0$ and $a_{Mm} \neq 0$. This implies that D has at most regular singularities at O and at ∞. Furthermore, it has (regular or irregular) singularities at the roots of the polynomial $\sum_{h=0}^{M} a_{hm} u^h$. Suppose this polynomial has N complex roots λ_i, of order m_i, $i = 1, ..., N$. Obviously,

$$\sum_{i=1}^{N} m_i = M$$

We shall use the following result (cf. [17]).

THEOREM 3.1. *Let* $D = \sum_{l=0}^{m} a_l(u)(d/du)^l$, *where* $a_l \in \mathbb{C}\{u\}$ *for* $l \in \{0, 1, ..., m\}$. *If* a_m *has a zero of (finite) order* n *at* O, *then* $\dim \ker(D, \mathscr{C}(0)) = n$.

We shall suppose that at each singular point λ_i ($i \in \{1, .., N\}$) a basis of microsolutions of (D) is given (cf. Remark 3.2). By virtue of Theorem 3.1, the total number of these microsolutions will be equal to M, the order of the difference equation $(\Delta') = (\mathscr{P}D)$. Using Mellin transforms of these microsolutions, we shall define fundamental systems of solutions of (Δ'). Each of these fundamental systems is characterized by a certain growth property.

In Subsection 3.1, we begin by discussing the global properties of solutions of (D). To this end, we take a fundamental system of solutions of (D)

at the origin. The singularity at λ_i of an element of the given fundamental system is determined by its coefficients ("connection coefficients") with respect to the given basis of microsolutions at λ_i. In Subsection 3.2, we define different fundamental systems of (Δ'). The relations between these fundamental systems are discussed in Subsection 3.3. They can be expressed in terms of the "connection coefficients" mentioned above. The section is concluded with two simple examples.

3.1. *Global Properties of* (D)

We assume that the zeroes λ_i ($i \in \{1, ..., N\}$) of $\sum_{h=0}^{M} a_{hm} u^h$ have been ordered in such a way that

$$\begin{gathered} 0 \leqslant \arg \lambda_i \leqslant \arg \lambda_{i+1} < 2\pi \\ |\lambda_i| \leqslant |\lambda_{i+1}| \qquad \text{whenever} \quad \arg \lambda_i = \arg \lambda_{i+1}. \end{gathered} \tag{3.1}$$

For all $i \in \{1, ..., N\}$ and all $k \in \mathbb{Z}$ let

$$\lambda_{i+kN} = \lambda_i e^{2k\pi i}. \tag{3.2}$$

Thus $\{\lambda_i, i \in \mathbb{Z}\}$ is the set of singular points of D on $\mathbb{C}_\infty$. For all $i \in \mathbb{Z}$ we put

$$\mathscr{C}(\lambda_i) = \mathscr{C}_i, \qquad \operatorname{can}_{(\lambda_i)} = \operatorname{can}_i, \qquad \operatorname{var}_{(\lambda_i)} = \operatorname{var}_i, \qquad T_{(\lambda_i)} = T_i.$$

Let Φ_{ip}, $p = 1, ..., m_i$, be a basis of $\operatorname{Ker}(D, \mathscr{C}_i)$ and let Φ_i denote the m_i-dimensional vector with components Φ_{ip}, $p = 1, ..., m_i$. Furthermore, let r_i be a sufficiently small positive number and $\tilde{\varphi}_i \in (\tilde{\mathcal{O}}(\lambda_i, r_i))^{m_i}$ such that $\operatorname{can}_i \tilde{\varphi}_i = \Phi_i$. $\tilde{\varphi}_i$ is an m_i-dimensional vector function with components φ_{ip}, $p = 1, ..., m_i$. Let $\varphi_i = \operatorname{var}_i \Phi_i = (T_i - I)\tilde{\varphi}_i$. As $D\tilde{\varphi}_i \in (\mathcal{O}(\lambda_i, r_i))^{m_i}$, we have $D\varphi_i = 0$. The microfunctions $\operatorname{can}_i T_i \tilde{\varphi}_{ip}$, $p = 1, ..., m_i$, form another basis of $\operatorname{Ker}(D, \mathscr{C}_i)$. Hence there exists an invertible $m_i \times m_i$ matrix M_i with the property that

$$\operatorname{can}_i T_i \tilde{\varphi}_i = M_i \Phi_i, \qquad i \in \mathbb{Z}. \tag{3.3}$$

For all $i \in \mathbb{Z}$ let $\tilde{\varphi}_i^*$ be defined by

$$\tilde{\varphi}_i^*(u) = \tilde{\varphi}_{i+N}(ue^{2\pi i}), \qquad u \in \tilde{D}(\lambda_i, r_{i+N}). \tag{3.4}$$

Obviously, $\tilde{\varphi}_i^* \in \tilde{\mathcal{O}}(\lambda_i, r_{i+N})$ and $D\tilde{\varphi}_i^* \in \mathcal{O}(\lambda_i, r_{i+N})$. Hence there exists an invertible $m_i \times m_i$ matrix M_i^0 such that

$$\operatorname{can}_i \tilde{\varphi}_i^* = M_i^0 \Phi_i. \tag{3.5}$$

φ_i can be continued analytically along any path in $\mathbb{C}_\infty$ which avoids the points λ_j, $j \in \mathbb{Z}$. Let l_i be a path from λ_i to O such that $\arg u = \arg \lambda_i$, $\arg(u - \lambda_i) = \arg \lambda_i + \pi$ everywhere on l_i, except in the vicinity of some $\lambda_j \in (0, \lambda_i)$, where it describes a small semicircle about λ_j in the positive sense. By φ_{i0}, we shall denote the branch of φ_i obtained by analytic continuation along l_i.

At the origin, (D) possesses m linearly independent solutions $\varphi_p^0 \in \tilde{\mathcal{O}}(0, r)$, $p = 1, \ldots, m$, if r is a sufficiently small positive number. Let φ^0 denote the m-dimensional vector with components φ_p^0, $p = 1, \ldots, m$, and let M be the $m \times m$ matrix defined by

$$\varphi^0(ue^{2\pi i}) = M\varphi^0(u). \tag{3.6}$$

As $D\varphi_i = 0$, there exists an $m_i \times m$ matrix C_{i0} such that

$$\varphi_{i0} = C_{i0}\varphi^0. \tag{3.7}$$

Let φ^{0i} denote the branch of φ^0 obtained by analytic continuation along l_i in the opposite direction (i.e., from O to λ_i). There exists an $m \times m_i$ matrix C^{0i} such that

$$\operatorname{can}_i \varphi^{0i} = C^{0i}\Phi_i \tag{3.8}$$

From (3.7), it follows that $\varphi_i = C_{i0}\varphi^{0i}$ and hence we deduce, with (3.3) and (3.8),

$$M_i - I = C_{i0}C^{0i}. \tag{3.9}$$

Now let $j \neq i$ and let φ_i^j denote the branch of φ_i obtained by analytic continuation from λ_i to O along l_i and from O to λ_j along l_j in opposite direction. With (3.7) and (3.8), we have

$$\operatorname{can}_j \varphi_i^j = C_i^j\Phi_j, \qquad \text{where} \quad C_i^j = C_{i0}C^{0j}. \tag{3.10}$$

Remark 3.2. A basis of microsolutions of (D) at λ_i can be constructed in the following manner (cf. [17]). Take a basis $\{f_1, \ldots, f_m\}$ of $\ker(D, \tilde{\mathcal{O}}(\lambda_i))$ and representatives $h_j \in \mathcal{O}(\lambda_i)$, $j = 1, \ldots, l$, of a basis of $\operatorname{coker}(D, \mathcal{O}(\lambda_i))$, where l denotes the dimension of $\operatorname{coker}(D, \mathcal{O}(\lambda_i))$, and let $h_j = Dg_j$, $g_j \in \tilde{\mathcal{O}}(\lambda_i)$, $j = 1, \ldots, l$. Then the set of microfunctions $\{\operatorname{can}_i f_1, \ldots, \operatorname{can}_i f_m, \operatorname{can}_i g_1, \ldots, \operatorname{can}_i g_l\}$ generates $\ker(D, \mathscr{C}(\lambda_i))$. In particular, the functions h_j $(j = 1, \ldots, l)$ can be chosen to be holomorphic in the entire complex plane. In that case, the basis of microsolutions obtained in this way will consist of functions that are holomorphic on the universal covering of $\mathbb{C} - \{0, \lambda_1, \ldots, \lambda_N\}$.

3.2. *Fundamental Systems of Solutions of* (Δ')

Next, we turn to the difference equation (Δ') obtained from (D) by the transformation $\mathscr{P}$ defined in the Introduction; i.e.,

$$\sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl}(x-h-1)^l \, y(x-h) = 0.$$

Let ε be a positive number such that $\varepsilon < |\arg(\lambda_j - \lambda_i) - \arg \lambda_i|$ for all i, j for which $\arg \lambda_i \neq \arg \lambda_j$. For all $i \in \mathbb{Z}$ we put

$$\arg \lambda_i = \alpha_i, \qquad \arg \lambda_i + \varepsilon = \alpha_i^+, \qquad \arg \lambda_i - \varepsilon = \alpha_i^- .$$

We define vector functions z_i^+, z_i^-, and $\tilde{z}_i$ by

$$z_i^{\pm}(x) = \mathscr{P}_{\gamma_{\lambda_i}, \alpha_i^{\pm}}(\Phi_i)(x), \tag{3.11}$$

and

$$\tilde{z}_i(x) = \mathscr{P}_{l_i}(\Phi_i)(x), \tag{3.12}$$

where l_i is the path from λ_i to O defined in Subsection 3.1.

For all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$ we define a vector function $z^{0\alpha}$ by

$$z^{0\alpha}(x) = \mathscr{P}_{\gamma_{0,\alpha}}(\varphi^0)(x). \tag{3.13}$$

By means of partial integration it can be verified that each of these vector functions satisfies the difference equation (Δ'). With the aid of Propositions 1.3 and 2.5, it is easily seen that they are meromorphic in $\mathbb{C}$.

Let

$$P = \{\rho_q - n : q \in \{1, ..., m\}, n \in \mathbb{N}\},$$

and

$$\Sigma = \{\sigma_q + n - 2 : q \in \{1, ..., m\}, n \in \mathbb{N}\},$$

where $\rho_1, ..., \rho_m$ and $\sigma_1, ..., \sigma_m$ are the characteristic exponents relative to the regular singularities of (D) at ∞ and at O, respectively. For all $i \in \mathbb{Z}$, z_i^+ and z_i^- are holomorphic in $\mathbb{C} - P$, and $\tilde{z}_i$ is holomorphic in $\mathbb{C} - \Sigma$. For all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$, $z^{0\alpha}$ is holomorphic in $\mathbb{C} - (P \cup \Sigma)$.

THEOREM 3.3. *For each* $i \in \mathbb{Z}$, *the vector functions* $\lambda_i^x z_i^-$, $\lambda_i^x z_i^+$ *and* $\lambda_i^x \tilde{z}_i$ *have subexponential growth as* $x \to \infty$, *in the sector* $S[-\pi/2, \pi/2 + \varepsilon]$, $S[-\pi/2 - \varepsilon, \pi/2]$ *and* $S[-3\pi/2, -\pi/2 + \varepsilon]$, *respectively, where* ε *is some positive number.*

Furthermore, let $\varphi_i^{\pm}$ *denote the branch of* φ_i *obtained by analytic continuation along* $\gamma_{\lambda_j, x_i^{\pm}}$. *Let* U *be a contour of the type used in* (1.3), *enclosing* P *and let* $\tilde{U}$ *be a similar contour (but reflected in the imaginary axis) enclosing* Σ. *We have*

$$\varphi_i^{\pm}(u) = \frac{1}{2\pi i} \int_U z_i^{\pm}(x)\, u^{x-1}\, dx, \qquad |u| > \max_j |\lambda_j|$$

$$\varphi_{i0}(u) = \frac{1}{2\pi i} \int_{\tilde{U}} \tilde{z}_i(x)\, u^{x-1}\, dx, \qquad |u| < \min_j |\lambda_j|.$$

Moreover, the sets of solutions $Z^{\pm} : \{z_{ip}^{\pm} : p \in \{1, ..., m_i\},\ i \in \{1, ..., N\}\}$ *and* $\tilde{Z} := \{\tilde{z}_{ip} : p \in \{1, ..., m_i\},\ i \in \{1, ..., N\}\}$ *are fundamental systems of solutions of* (Δ').

Proof. All statements, except for the last one, follow immediately from Proposition 2.5. Now consider the set of solutions Z^-. There exists a fundamental system of solutions $\{\zeta_{ip}, p \in \{1, ..., m_i\}, i \in \{1, ..., N\}\}$ such that $\lambda_i^x \zeta_{ip}$ has subexponential growth as $x \to \infty$, in $S[-\pi/2, \pi/2 + \varepsilon]$ for some positive number ε (cf. [10, 5]). According to Proposition 2.5, each ζ_{ip} defines a microfunction $\Psi_{ip} \in \mathrm{Ker}(D, \mathscr{C}_i)$ such that $\mathscr{P}_{\gamma_{\lambda_i, x_i^-}}(\Psi_{ip})(x) = \zeta_{ip}(x)$. As the ζ_{ip} are linearly independent, the microsolutions Ψ_{ip} form a basis of $\mathrm{Ker}(D, \mathscr{C}_i)$ for each $i \in \{1, ..., N\}$. Hence, there exists an invertible $m_i \times m_i$ matrix C^i such that $\Phi_{ip} = \sum_{q=1}^{m_i} C^i_{pq} \Psi_{iq}$ and, consequently, $z_{ip}^- = \sum_{q=1}^{m_i} C^i_{pq} \zeta_{iq}$. This implies that Z^- is a fundamental system of solutions of (Δ'). In a similar manner, it can be proved that the same is true of Z^+ and $\tilde{Z}$. ∎

Remark 3.4. The price to be paid for the greater simplicity of the microsolution-approach as compared to the classical one is a loss of information about the asymptotic behaviour of the fundamental system of (Δ'). By the latter approach, one usually obtains solutions of (Δ') admitting a full asymptotic representation as $x \to \infty$ in a right or a left half-plane. This could also be achieved by a judicious choice of the microsolutions, but the difficulties involved are of the same order as those associated with the choice of appropriate paths of integration in the classical method.

Remark 3.5. Suppose that the precise form of the singularities of $z_i^{\pm}$ ($\tilde{z}_i$) is known. Then the integral representations of $\varphi_i^{\pm}$ (φ_{i0}) provided by Theorem 3.3 can be used to determine the behaviour of that specific branch of φ_i at ∞ (O), by means of residue calculus.

3.3. *Relations between the Fundamental Systems of Solutions of* (Δ')

We now proceed to examine the relations existing between the solutions of (Δ') defined above.The main results are stated in Theorem 3.6 below.

Let $i \in \mathbb{Z}$ and let $\zeta_i \in \{z_i^-, z_i^+, \tilde{z}_i\}$. From (3.4) and (3.5), we deduce the identity

$$\zeta_{i+N}(x) = e^{-2\pi i x} M_i^0 \zeta_i(x), \tag{3.14}$$

and similarly, from (3.6),

$$z^{0\alpha+2\pi}(x) = e^{-2\pi i x} M z^{0\alpha}(x), \qquad \alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\} \tag{3.15}$$

By deformation of contours, it is easily verified that, for sufficiently large values of Re x,

$$z_i^-(x) - z_i^+(x) = \sum_{j > i \,:\, \alpha_j = \alpha_i} \mathscr{P}_{\Gamma_{\lambda_j, \alpha_j^-}}(\varphi_i^j)(x),$$

where $\Gamma_{\lambda_j, \alpha_j^-}$ is defined in Remark 2.4 and $\mathscr{P}_{\Gamma_{\lambda_j, \alpha_j^-}}(\varphi_i^j)(x) = \mathscr{M}_{\Gamma_{\lambda_j, \alpha_j^-}}(\varphi_i^j)(1-x)$. Hence it follows, with (2.1) and (3.10), that

$$z_i^- - z_i^+ = \sum_{j > i \,:\, \alpha_j = \alpha_i} C_i^j z_j^- . \tag{3.16}$$

In a similar manner, we find, with (3.8),

$$z^{0\alpha_i^-} - z^{0\alpha_i^+} = \sum_{j \,:\, \alpha_j = \alpha_i} C^{0j} z_j^- . \tag{3.17}$$

From (3.6), we further deduce that, for sufficiently large values of Re x and $|\text{Im}\, x|$,

$$\mathscr{P}_{\Gamma_{0, \alpha_i^-}}(\varphi^0)(x) = (e^{-2\pi i x} M - I)\, z^{0\alpha_i^-}. \tag{3.18}$$

On the other hand, by deforming the contour Γ_{0, α_i^-}, we find

$$\mathscr{P}_{\Gamma_{0, \alpha_i^-}}(\varphi^0)(x) = - \sum_{j \,:\, \alpha_i \leqslant \alpha_j < \alpha_i + 2\pi} \mathscr{P}_{\Gamma_{\lambda_j, \alpha_j^-}}(\varphi^{0j})(x)$$

and hence, with (3.8) and (3.11),

$$\mathscr{P}_{\Gamma_{0, \alpha_i^-}}(\varphi^0)(x) = - \sum_{j \,:\, \alpha_i \leqslant \alpha_j < \alpha_i + 2\pi} C^{0j} z_j^- . \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$z^{0\alpha_i^-} = (I - e^{-2\pi i x} M)^{-1} \sum_{j \,:\, \alpha_i \leqslant \alpha_j < \alpha_i + 2\pi} C^{0j} z_j^- . \tag{3.20}$$

Furthermore we have, for sufficiently large values of Re x and $|\text{Im}\, x|$,

$$\tilde{z}_i(x) - z_i^-(x) + \mathscr{P}_{\gamma_{0, \alpha_i^- + 2\pi}}(\varphi_{i0})(x) = - \sum_{j > i \,:\, \alpha_j < \alpha_i + 2\pi} \mathscr{P}_{\Gamma_{\lambda_j, \alpha_j^-}}(\varphi_i^j)(x).$$

Hence it follows, with (2.3), (3.7), and (3.10), that

$$\tilde{z}_i - z_i = -C_{i0} z^{0\alpha_{i+N}^{-}} - \sum_{j>i:\,\alpha_j<\alpha_i+2\pi} C_i^j z_j^- .$$

Using (3.15) and (3.20), we find

$$\tilde{z}_i - z_i^- = -e^{-2\pi i x} C_{i0} M(I - e^{-2\pi i x} M)^{-1}$$
$$\times \sum_{j:\,\alpha_i\leqslant\alpha_j<\alpha_i+2\pi} C^{0j} z_j^- - \sum_{j>i:\,\alpha_j<\alpha_i+2\pi} C_i^j z_j^- .$$

Noting that $C_i^j = C_{i0} C^{0j}$ and using (3.9), we finally obtain the following result.

THEOREM 3.6. *For all $i \in \mathbb{Z}$ we have*

$$z_i^- - z_i^+ = \sum_{j>i:\,\alpha_j=\alpha_i} C_i^j z_j^- .$$

and

$$\tilde{z}_i = \{M_i - C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0i}\} z_i^-$$
$$+ \sum_{j<i:\,\alpha_j=\alpha_i} \{C_i^j - C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0j}\} z_j^-$$
$$- \sum_{j>i:\,\alpha_j<\alpha_i+2\pi} C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0j} z_j^- .$$

Remark 3.7. With the aid of (3.14), the relations between the fundamental systems Z^-, Z^+ and $\tilde{Z}$ now follow immediately. For instance, assuming that $|\lambda_1| = \min_{j:\,\alpha_j=\alpha_1} |\lambda_j|$, we have, for all $i \in \{1, ..., N\}$,

$$\tilde{z}_i = \{M_i - C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0i}\} z_i$$
$$- \sum_{j=i+1}^{N} C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0j} z_j^-$$
$$+ \sum_{j<i:\,\alpha_j=\alpha_i} \{C_i^j - C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0j}\} z_j^- - e^{-2\pi i x}$$
$$\times \sum_{j\geqslant 1:\,\alpha_j<\alpha_i} C_{i0}(I - e^{-2\pi i x} M)^{-1} C^{0j+N} M_j^0 z_j^- \tag{3.21}$$

Remark 3.8. By the use of Mellin transforms of Φ_i over more complicated types of contours, one may define solutions y_i of (Δ') with the property that $\lambda_i^x z_i(x)$ has subexponential growth in sectors different from those mentioned in Theorem 3.3 (cf. Remark 2.6). For example, let $\varepsilon > 0$

and let $\lambda_i^{\pm} = \lambda_i(1 \pm \varepsilon)$. If ε is sufficiently small, the vector function $z_{i,\pm}$ defined by

$$z_{i,\pm}(x) = \int_{C_{\lambda_i, \lambda_i^{\pm}}} \tilde{\varphi}_i(u)\, u^{-x} du + \int_{\Gamma^+_{\lambda_i^{\pm}}} \varphi_i(u)\, u^{-x}\, du,$$

where $\Gamma^+_{\lambda_i^{\pm}}$ is defined by (1.2), is a solution of (Δ'). Moreover, $\lambda_i^x z_{i,\pm}(x)$ has subexponential growth as $x \to \infty$ in $S(-\pi, 0)$ (uniformly on the half plane $\operatorname{Im} x < -K$ for some $K > 0$). It is easily seen that there exist $m_i \times m_j$ matrices $\tilde{C}_i^j$ such that

$$z_{i,+} - z_{i,-} = \sum_{j > i:\, |\lambda_j| = |\lambda_i|} \tilde{C}_i^j z_{j,-}.$$

EXAMPLE 3.9. $D = (u-1)^2 u(d/du) - u$. The corresponding difference equation $(\Delta') = (\mathscr{P}D)$ is the hypergeometric difference equation

$$(x-3)\, y(x-2) - (2x-3)\, y(x-1) + (x-1)\, y(x) = 0. \qquad (\Delta')$$

(D) has an irregular singularity of level 1 at 1 and is regular throughout the rest of the extended complex plane. Obviously, $N = 1$ and $m_1 = 2$. Let

$$\tilde{\varphi}_1(u) = e^{-1/(u-1)}, \qquad \tilde{\varphi}_2(u) = \frac{1}{2\pi i} \int_0^{-\infty} (t-1)^{-1} e^{-t/(u-1)}\, dt$$

$$\left(\frac{\pi}{2} < \arg(u-1) < \frac{3\pi}{2}\right).$$

Here, and in what follows, we drop the subscript 1, i.e.: $\tilde{\varphi}_1 = \tilde{\varphi}_{11}$, $\tilde{\varphi}_2 = \tilde{\varphi}_{12}$, etc. We have

$$D\tilde{\varphi}_1(u) = 0, \qquad D\tilde{\varphi}_2(u) = \frac{u(u-1)}{2\pi i}.$$

$\tilde{\varphi}_2$ can be continued analytically to $\tilde{D}(1, 1)$ and the microfunctions Φ_1 and Φ_2 defined by

$$\Phi_1 = \operatorname{can}_1 \tilde{\varphi}_1, \qquad \Phi_2 = \operatorname{can}_1 \tilde{\varphi}_2$$

form a basis of $\operatorname{Ker}(D, \mathscr{C}(1))$ (cf. [17] and Remark 3.2). It is easily verified that

$$\varphi_1 = 0, \qquad \varphi_2 = \tilde{\varphi}_1 \qquad (3.22)$$

and hence

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Furthermore, we choose

$$\varphi^0 = e^{-1/(u-1)} = \tilde{\varphi}_1.$$

Then we have, from (3.6)–(3.8),

$$M = (1), \qquad C_{10} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C^{01} = (1 \;\; 0).$$

Next we turn to the difference equation (Δ'). With (3.22), we find

$$z_1^+ = z_1^- = \int_C e^{-1/(u-1)} u^{-x} \, du$$

where C is a circular path about 1, described in the positive sense, and

$$z_2^+ = z_2^- = \int_1^\infty e^{-1/(u-1)} u^{-x} \, du.$$

Furthermore, we have $\tilde{z}_1 = z_1^-$ and

$$\tilde{z}_2 = \int_{C_{1,\mu}} \tilde{\varphi}_2(u) \, u^{-x} \, du + \int_\mu^0 e^{-1/(u-1)} u^{-x} \, du \tag{3.23}$$

where $\mu = 1 + \varepsilon e^{i\pi}$ $(0 < \varepsilon < 1)$ and $C_{1,\mu}$ is defined in Definition 2.2. Now $\tilde{\varphi}_2$ can be shown to be bounded on the sector $0 \leqslant \arg(u-1) \leqslant 2\pi$, $|u-1| \leqslant 1$. Hence it follows that

$$\int_{C_{1,1+\varepsilon}} \tilde{\varphi}_2(u) \, u^{-x} \, du = \int_1^{1+\varepsilon} \varphi_2(u) \, u^{-x} \, du$$

Combining this with (3.23), we get

$$\tilde{z}_2 = \int_l e^{-1/(u-1)} u^{-x} \, du,$$

where l is the path consisting of the segment from 1 to $1+\varepsilon$, the arc of $C_{1,1+\varepsilon}$ from $1+\varepsilon$ to μ, and the segment from μ to O. Obviously,

$$\tilde{z}_2 - z_2^- = -z^{0\alpha} = z_1^- - z^{0\beta}, \qquad \text{with} \quad -2\pi < \beta < 0 < \alpha < 2\pi$$

Moreover,

$$z^{0\beta} = (1 - e^{-2\pi i x})^{-1} z_1^-.$$

Hence

$$\tilde{z}_2 - z_2^- = (1 - e^{2\pi i x})^{-1} z_1^-,$$

and this identity is in agreement with the result stated in Theorem 3.6.

EXAMPLE 3.10. D is of Fuchsian type. In this case we may choose $\tilde{\varphi}_i \in \tilde{\mathscr{C}}(\lambda_i, r_i)^{m_i}$ of the form

$$\tilde{\varphi}_{ip}(u) = (u-\lambda_i)^{\rho_{ip}} f_{ip}(u), \tag{3.24}$$

where $\rho_{ip} \in \mathbb{C}$ and $f_{ip} \in \mathbb{C}\{u-\lambda_i\}[\log(u-\lambda_i)]$ for all $p \in \{1, ..., m_i\}$ and all $i \in \mathbb{Z}$. Moreover, we may take $\rho_{i+Np} = \rho_{ip}$ and

$$M_i^0 = M_i \tag{3.25}$$

for all $i \in \mathbb{Z}$. The difference equation $(\Delta') := (\mathscr{P}D)$ is sometimes called a regular difference equation. It is a well-known fact that the solutions z_{ip}^-, z_{ip}^+ and $\tilde{z}_{ip}$ defined by (3.11) and (3.12) admit an asymptotic representation of the form

$$\hat{z}_{ip}(x) = \lambda_i^{-x} x^{-\rho_{ip}-1} h_{ip}(x),$$

as $x \to \infty$ in the sectors $S(-\pi/2, \pi/2+\varepsilon)$, $S(-\pi/2-\varepsilon, \pi/2)$ and $S(-3\pi/2-\varepsilon, -\pi/2)$, respectively, where $h_{ip} \in \mathbb{C}[\![x^{-1}]\!][\log x]$ and ε is some positive number (cf. [18]).

4. THE EQUATIONS (D_1) AND (Δ_1)

Let D be a differential operator of the type considered in the previous section, i.e.,

$$D = \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} u^h \left(u \frac{d}{du}\right)^l$$

with $a_{0m} \neq 0$ and $a_{Mm} \neq 0$. By the formal Laplace transformation (0.4), D is transformed into the differential operator

$$D_1 := \sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} (-1)^{h+l} \left(\frac{d}{dt}\right)^h \left(\frac{d}{dt} t\right)^l.$$

D_1 is a differential operator of order $m+M$, with two singularities: a regular one at O and an irregular one at ∞. Its Newton polygon at ∞ has slopes equal to 0 and 1 (cf. [5]). At ∞, the corresponding homogeneous differential equation (D_1) possesses a formal fundamental system of solutions of the form

$$\begin{aligned} \hat{\psi}_{ip}(t) &= \exp\{-\lambda_i t + q_{ip}(t)\}\, t^{-\rho_{ip}-1} g_{ip}(t), \\ & \qquad p \in \{1, ..., m_i\}, \qquad i \in \{1, ..., N\} \\ \hat{\psi}_p^0(t) &= t^{-\rho_p^0 - 1} g_p^0(t), \qquad p \in \{1, ..., m\} \end{aligned} \tag{4.1}$$

where q_{ip} is a polynomial in $t^{1/k}$ for some $k \in \mathbb{N}$, of degree less than k, $\rho_{ip} \in \mathbb{C}$, $\rho_p^0 \in \mathbb{C}$, $g_{ip} \in \mathbb{C}[[t^{-1/k}]][\log t]$, and $g_p^0 \in \mathbb{C}[[t^{-1}]][\log t]$.

Taking suitable Laplace transforms of micro-solutions of (D), in Subsection 4.1, we obtain solutions of (D_1) with particular asymptotic properties at ∞. These solutions present a Stokes phenomenon that can be described in terms of the connection matrices C_{i0} and C^{i0} defined in Section 3. In Subsection 4.2, we define fundamental systems of the difference equation $(\Delta_1) := (\mathcal{M} D_1)$, using Mellin transforms of solutions of (D_1). Each of these fundamental systems has a simple asymptotic behaviour in either a right or a left half- plane. We discuss the relation between the different fundamental systems of (Δ_1), as well as their relation to the fundamental systems of the difference equation $(\mathcal{P} D)$, obtained by the method of Section 3.

We use the same notations and conventions as in the previous sections. We recall that $\gamma_{\lambda, \alpha}$ denotes the half line from λ to ∞ in the direction $\bar{\alpha}$ and $\Gamma_{\lambda, \alpha}$ is the contour defined in Remark 2.4.

4.1. *Solutions of* (D_1) *and the Stokes Phenomenon at* ∞

Let $i \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ such that $\alpha \neq \arg(\lambda_j - \lambda_i)$ for all $j \in \mathbb{Z} : j \neq i$. The vector function ψ_i^α defined by

$$\psi_i^\alpha(t) = \mathcal{L}_{\gamma_{\lambda_i, \alpha}}(\Phi_i)(t), \qquad t \in S\left(-\alpha - \frac{\pi}{2}, -\alpha + \frac{\pi}{2}\right),$$

is a solution of the differential equation (D_1). It can be continued analytically to $\mathbb{C}_\infty$. Furthermore, it has particular asymptotic properties in a slightly larger sector than $S(-\alpha - \pi/2, -\alpha + \pi/2)$:

PROPOSITION 4.1 (cf. [17, 16, 15]). *Let $i \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ such that $\alpha \neq \arg(\lambda_j - \lambda_i)$ for all $j \in \mathbb{Z}$, $j \neq i$. Then the mapping $\mathcal{L}_{\gamma_{\lambda_i, \alpha}}$ is a bijection from the space of microsolutions of (D) at λ_i onto the space of solutions ψ of (D_1) with the property that $\psi(t) \exp(\lambda_i t)$ has subexponential growth as $t \to \infty$ in the sector $S(-\alpha - \pi/2 - \delta, -\alpha + \pi/2 + \delta)$, for some $\delta > 0$.*

Remark 4.2. For a given $\alpha \in \mathbb{R}$, the basis of microsolutions $\{\Phi_{ip}, p = 1, ..., m_i\}$, can be chosen in such a way that, for all $p \in \{1, ..., m_i\}$, $\mathcal{L}_{\gamma_{\lambda_i, \alpha}}(\Phi_{ip})$ admits an asymptotic representation of the form $\hat{\psi}_{ip}$ as $t \to \infty$ in the sector mentioned in Proposition 4.1. In general, this basis will depend on α.

Now, let $\beta \in (-\alpha - \pi/2 - \delta, -\alpha + \pi/2 + \delta)$ such that $\mathrm{Re}(\lambda_i e^{i\beta}) > 0$. Then, according to Proposition 4.1, $\psi_i^\alpha(t)$ decreases exponentially as $t \to \infty$ on $\gamma_{0, \beta}$. Due to the regular singularity of (D_1) at the origin, $\psi_i^\alpha(t)$ admits an asymptotic expansion of the type (1.1) (with $\lim_{n \to \infty} \mathrm{Re}\, \rho_n = \infty$) as $t \to 0$ on $\gamma_{0, \beta}$. Therefore, $\mathcal{M}_{\gamma_{0, \beta}}(\psi_i^\alpha)$ exists and, by Proposition 1.3, is analytic in a right half-plane. By means of partial integration it is easily verified that $\mathcal{M}_{\gamma_{0,\beta}}(\psi_i^\alpha)$ satisfies the difference equation $(\Delta_1) := (\mathcal{M}D_1)$. However, in general this solution cannot be characterized in a simple way by its asymptotic behaviour. In this subsection we introduce particular solutions of (D_1), which give rise to solutions of (Δ_1) that do have simple asymptotic properties and are closely related to the solutions $z_i^\pm$ and $\tilde{z}_i$ of (Δ') in Section 3.

For all $i \in \mathbb{Z}$ we define

$$\psi_i^\pm(t) = \mathcal{L}_{\gamma_{\lambda_i, \alpha_i^\pm}}(\Phi_i)(t), \qquad t \in S\left(-\frac{\pi}{2} - \alpha_i^\pm, \frac{\pi}{2} - \alpha_i^\pm\right), \tag{4.2}$$

$$\tilde{\psi}_i(t) = \mathcal{L}_{\gamma_{\lambda_i, \alpha_i^- + \pi}}(\Phi_i)(t), \qquad t \in S\left(-\frac{3\pi}{2} - \alpha_i^-, -\frac{\pi}{2} - \alpha_i^-\right).$$

Furthermore, for any $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$, we define

$$\psi^{0\alpha}(t) = \mathcal{L}_{\Gamma_{0,\alpha}}(\varphi^0)(t), \qquad t \in S\left(-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right). \tag{4.3}$$

Note that $\psi_p^{0\alpha} \equiv 0$ if φ_p^0 is regular at O. For all $p \in \{1, ..., m\}$ for which φ_p^0 is not regular at O, $\psi_p^{0\alpha}$ admits an asymptotic expansion of the type (1.1), as $t \to \infty$ in $S(-\pi/2 - \alpha - \delta, \pi/2 - \alpha + \delta)$ for some $\delta > 0$. With the aid of (3.6) it is easily verified, by deformation of contours, that for all $i \in \mathbb{Z}$ and for all $t \in S(-\pi/2 - \alpha_i, \pi/2 - \alpha_i)$,

$$\psi^{0\alpha_i^-}(t) - \psi^{0\alpha_i^+}(t) = (M - I) \sum_{j : \alpha_j = \alpha_i} \int_{\Gamma_{\lambda_j, \alpha_j^-}} \varphi^{0j}(u)\, e^{-ut}\, du.$$

With (2.1), (3.8), and (4.2), it follows that

$$\psi^{0\alpha_i^-}(t) - \psi^{0\alpha_i^+}(t) = (M - I) \sum_{j : \alpha_j = \alpha_i} C^{0j} \psi_j^- . \tag{4.4}$$

Similarly, using (3.10), we obtain the identities

$$\psi_i^- - \psi_i^+ = \sum_{j > i : \alpha_j = \alpha_i} C_i^j \psi_j^- \tag{4.5}$$

and

$$\tilde{\psi}_i = \psi_i^- - \sum_{j > i : \alpha_j < \alpha_i + \pi} C_i^j \psi_j^- . \tag{4.6}$$

Due to (3.3), we have, for all $j \in \mathbb{Z}$,

$$\mathcal{L}_{\gamma_{\lambda_j, \alpha_j^\pm + 2\pi}}(\Phi_j) = M_j \mathcal{L}_{\gamma_{\lambda_j, \alpha_j^\pm}}(\Phi_j) \tag{4.7}$$

From (3.4), (3.5), and (4.7), we deduce that, for all $t \in S(-5\pi/2-\alpha_j^{\pm}, -3\pi/2-\alpha_j^{\pm})$,

$$\psi_{j+N}^{\pm}(t)=M_j^0 \mathscr{L}_{\gamma_{\lambda_j,\alpha_j^{\pm}}}(\Phi_j)(t)=M_j^0 M_j^{-1} \mathscr{L}_{\gamma_{\lambda_j,\alpha_j^{\pm}+2\pi}}(\Phi_j)(t). \tag{4.8}$$

Similarly, in view of (3.6), we have, for all $\alpha \in \mathbb{R}-\{\alpha_i : i \in \mathbb{Z}\}$

$$\mathscr{L}_{\Gamma_{0,\alpha+2\pi}}(\varphi^o)=M\mathscr{L}_{\Gamma_{0,\alpha}}(\varphi^o) \tag{4.9}$$

and

$$\psi^{0\alpha+2\pi}(t)=M\mathscr{L}_{\Gamma_{0,\alpha}}(\varphi^o)(t), \qquad t \in S\left(-\frac{5\pi}{2}-\alpha, -\frac{3\pi}{2}-\alpha\right). \tag{4.10}$$

By deformation of contours we find, for all $t \in S(-3\pi/2-\alpha_i, -3\pi/2-\alpha_i^-)$,

$$\begin{aligned}\tilde{\psi}_i(t)&-\mathscr{L}_{\gamma_{\lambda_i,\alpha_i^-+2\pi}}(\Phi_i)(t)\\ &=\sum_{j:\alpha_i+\pi\leqslant\alpha_j<\alpha_i+2\pi}\int_{\Gamma_{\lambda_j,\alpha_j^-}}\varphi_i^j(u)\,e^{-ut}\,du+\int_{\Gamma_{0,\alpha_i^-}}\varphi_{i0}(u)\,e^{-ut}\,du\\ &\quad+\sum_{j<i:\alpha_j=\alpha_i}\int_{\Gamma_{\lambda_j,\alpha_j^-}}\varphi_i^j(u)\,e^{-ut}\,du.\end{aligned}$$

With (2.1), (3.7), and (4.7)–(4.10), it follows that

$$\begin{aligned}\tilde{\psi}_i=M_i(M_i^0)^{-1}\psi_{i+N}^-&+\sum_{j:\alpha_i+\pi\leqslant\alpha_j<\alpha_i+2\pi}C_i^j\psi_j^-+C_{i0}M^{-1}\psi^{0\alpha_{i+N}^-}\\ &+\sum_{j<i+N:\alpha_j=\alpha_i}C_i^j(M_j^0)^{-1}\psi_{j+N}^-.\end{aligned} \tag{4.11}$$

The coefficients of the solutions of (D_1) on the right-hand sides of formulas (4.6) and (4.11) are Stokes multipliers of level 1. They determine the change in the type of exponential growth or decrease (of order 1) of $\tilde{\psi}_i$ as the direction in which $t\to\infty$ passes certain Stokes directions of level 1. (The Stokes directions of level 1 are given by the numbers $\pi/2-\arg(\lambda_j-\lambda_i)$, $i, j\in\mathbb{Z}$, $i\neq j$, and $\pm\pi/2-\arg\lambda_j$, $j\in\mathbb{Z}$.) For example, while ψ_i^- is the dominating term in (4.6) and hence determines the asymptotic behaviour of $\tilde{\psi}_i$ when $\arg t+\pi/2+\alpha_i$ is sufficiently small, it becomes subdominant as soon as $\arg t$ is increased to a value $>-\pi/2-\arg(\lambda_i-\lambda_j)$ for some j such that $\alpha_j<\alpha_i+\pi$ and $C_i^j\neq 0$.

A more refined description of the asymptotic properties of solutions of (D_1) can be obtained by choosing, for each $p\in\{1, ..., m_i\}$, solutions $\psi_{ip}^{\pm}$ and $\tilde{\psi}_{ip}$ that are represented asymptotically by the formal solution $\hat{\psi}_{ip}$ (cf. (4.1)). In general this can only be achieved by using two different bases of microsolutions of (D) in the definitions of these functions, given in (4.2) (cf. Remark 4.2). In that case the connection formulas (4.6) and (4.11) will have to be modified to include extra terms originating from the corresponding basis transformation. These extra terms describe the Stokes phenomenon associated with lower levels (i.e., the numbers $(1/k)\deg(q_{ip}-q_{ir})$, $p, r \in \{1, ..., m_i\}$, $i \in \mathbb{Z}$), which is related to a change in exponential behaviour of order <1.

4.2. *Fundamental Systems of Solutions of* (Δ_1) *and Connection Formulas*

Now we are able to define the solutions of (Δ_1) announced above. From Proposition 4.1, we infer that, for all $i \in \mathbb{Z}$, $\psi_i^\pm$ decreases exponentially as $t \to \infty$ in $S(-\alpha_i - \pi/2, -\alpha_i + \pi/2)$. Furthermore, for all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$, $\psi^{0\alpha}$ has moderate growth as $t \to \infty$ in $S(-\alpha - \pi/2, -\alpha + \pi/2)$. For all $i \in \mathbb{Z}$, we define

$$y_i^\pm = \mathcal{M}_{\gamma_{0,\beta}}(\psi_i^\pm), \qquad \beta \in \left(-\alpha_i - \frac{\pi}{2}, -\alpha_i + \frac{\pi}{2}\right), \tag{4.12}$$

and, for all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$,

$$y^{0\alpha} = \mathcal{M}_{\gamma_{0,\beta}}(\psi^{0\alpha}), \qquad \beta \in \left(-\alpha - \frac{\pi}{2}, -\alpha + \frac{\pi}{2}\right), \tag{4.13}$$

By means of partial integration it can be verified that each of these vector functions satisfies the difference equation (Δ_1).

From (4.8) and (4.12), we derive the identity

$$y_{i+N}^\pm(x) = e^{-2\pi i x} M_i^0 y_i^\pm(x), \qquad i \in \mathbb{Z}. \tag{4.14}$$

Similarly, from (4.10) and (4.13), we deduce

$$y^{0\alpha+2\pi}(x) = e^{-2\pi i x} M y^{0\alpha}(x), \qquad \alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}. \tag{4.15}$$

Using (4.4), (4.12), and (4.13), we find

$$y^{0\alpha_i^-} - y^{0\alpha_i^+} = (M - I) \sum_{j : \alpha_j = \alpha_i} C^{0j} y_j^-, \qquad i \in \mathbb{Z}.$$

Hence it follows that, for all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$,

$$y^{0\alpha} - y^{0\alpha+2\pi} = (M - I) \sum_{j : \alpha < \alpha_j < \alpha + 2\pi} C^{0j} y_j^-.$$

With (4.15), we obtain the identity

$$(I - e^{-2\pi i x} M)\, y^{0\alpha} = (M - I) \sum_{j : \alpha < \alpha_j < \alpha + 2\pi} C^{0j} y_j^-. \tag{4.16}$$

PROPOSITION 4.3. *For all $i \in \mathbb{Z}$ and all $\alpha \in \mathbb{R} - \{\alpha_i : i \in \mathbb{Z}\}$, the following identities hold*:

$$y_i^{\pm}(x) = \Gamma(x)\, z_i^{\pm}(x), \qquad y^{0\alpha}(x) = \Gamma(x)(M-I)\, z^{0\alpha}(x).$$

Here $z_i^{\pm}$ and $z^{0\alpha}$ are the solutions of $(\Delta') := (\mathscr{P}D)$ defined in Section 3.

Proof. The statement concerning $y_i^{\pm}$ can be easily proved by the use of integral representations in appropriate regions of $\mathbb{C}$. For example, for sufficiently large values of Re x we have

$$y_i^{+}(x) = \int_0^{\infty e^{i\beta}} \psi_i^{+}(t)\, t^{x-1}\, dt, \tag{4.17}$$

where $\beta \in (-\alpha_i - \pi/2, -\alpha_i + \pi/2)$. Furthermore, $\psi_i^{+} = \mathscr{L}_{\gamma_{\lambda_i, \alpha_i^{+}}}(\Phi_i)$ and can be represented in the following way (cf. Definition 2.2):

$$\psi_i^{+}(t) = \int_{C_{\lambda_i, \lambda_i'}} \tilde{\varphi}_i(u)\, e^{-ut}\, du + \int_{\gamma_{\lambda_i', \alpha_i^{+}}} \varphi_i(u)\, e^{-ut}\, du.$$

Inserting this into (4.17) and reversing the order of integration, one obtains the desired result. The argument is essentially the same as the one used in the proof of Theorem 2.2.5 in [5].

The last statement now follows from (3.20) and (4.16). ∎

COROLLARY 4.4. *The sets of functions $Y^{\pm} := \{y_{ip}^{\pm} : p \in \{1, ..., m_i\}, i \in \{1, ..., N\}\}$ are fundamental systems of solutions of (Δ_1), analytic in a right half-plane, with the following asymptotic property. There exists a positive number δ such that, for all $i \in \{1, ..., N\}$ $x^{-x}(e\lambda_i)^x\, y_i^{+}(x)$ has subexponential growth as $x \to \infty$ in $S[-\pi/2 - \delta, \pi/2]$ and $x^{-x}(e\lambda_i)^x\, y_i^{-}(x)$ has subexponential growth as $x \to \infty$ in $S[-\pi/2, \pi/2 + \delta]$.*

Corollary 4.4 follows immediately from Theorem 3.3, Proposition 4.3, and Stirling's formula.

Before we define a fundamental system of solutions of (Δ_1) that is analytic in a left half-plane, we begin by considering the very simple example, already mentioned in the introduction, of the differential equation

$$t\frac{d}{dt}\psi(t) + t\psi(t) = 0 \tag{D_1}$$

and corresponding difference equation

$$y(x+1) - xy(x) = 0. \tag{Δ_1}$$

The function $\psi(t) := e^{-t}$ is a solution of (D_1) and its Mellin transform $y(x) := \mathscr{M}_{\gamma_{0,0}}(\psi) = \Gamma(x)$ is a solution of (Δ_1), analytic in a right half-plane.

A solution of (Δ_1) that is analytic in a left half- plane can be obtained by integrating $e^{-t}t^{x-1}$ over a suitable contour. For example, we may take

$$\tilde{y}(x) := \mathcal{M}_{\gamma_{0,-\pi/2}}(\psi)(x) - \mathcal{M}_{\gamma_{0,-3\pi/2}}(\psi)(x) = (1 - e^{-2\pi i x})\,\Gamma(x)$$

(note that the Mellin transforms $\mathcal{M}_{\gamma_{0,-\pi/2}}(\psi)(x)$ and $\mathcal{M}_{\gamma_{0,-3\pi/2}}(\psi)(x)$ exist in the sense of Definition 1.5). Furthermore, $x^{-x}e^{x}\tilde{y}(x)$ has moderate growth as $x \to \infty$ in the sector $S(-2\pi+\delta, -\delta)$ for every $\delta > 0$.

Now let us first suppose that, for all $i \in \mathbb{Z}$, ψ_i^- has moderate growth as $t \to \infty$ in the sector $S[-\alpha_i - \pi/2, -\alpha_i + \pi/2]$. (This is the case if, for example, (D) is a Fuchsian differential equation.) From (4.6) and the fact that

$$\gamma_{0,-\alpha_i-\pi/2} \subset \bigcap_{j>i:\,\alpha_j<\alpha_i+\pi} S\left[-\alpha_j - \frac{\pi}{2}, -\alpha_j + \frac{\pi}{2}\right],$$

it follows that $\tilde{\psi}_i$ has moderate growth as $t \to \infty$ on $\gamma_{0,-\alpha_i-\pi/2}$. Consequently, $\mathcal{M}_{\gamma_{0,-\alpha_i-\pi/2}}(\tilde{\psi}_i)$ exists and, moreover, we have

$$\mathcal{M}_{\gamma_{0,-\alpha_i-\pi/2}}(\tilde{\psi}_i) = y_i^- - \sum_{j>i:\,\alpha_j<\alpha_i+\pi} C_i^j y_j^- . \tag{4.18}$$

Similarly, using (4.11) and noting that

$$\gamma_{0,-\alpha_i-3\pi/2} \subset \bigcap_{j:\,\alpha_i+\pi\leqslant\alpha_j\leqslant\alpha_i+2\pi} S\left[-\alpha_j - \frac{\pi}{2}, -\alpha_j + \frac{\pi}{2}\right],$$

we conclude that $\mathcal{M}_{\gamma_{0,-\alpha_i-3\pi/2}}(\tilde{\psi}_i)$ exists and, furthermore, that the following identity holds:

$$\begin{aligned}\mathcal{M}_{\gamma_{0,-\alpha_i-3\pi/2}}(\tilde{\psi}_i) = M_i(M_i^0)^{-1}\, y_{i+N}^- + \sum_{j:\,\alpha_i+\pi\leqslant\alpha_j<\alpha_{i+N}} C_i^j y_j^- + C_{i0} M^{-1} y^{0\alpha_{i+N}^-} \\ + \sum_{j<i:\,\alpha_j=\alpha_i} C_i^j (M_j^0)^{-1}\, y_{j+N}^-\end{aligned} \tag{4.19}$$

For all $i \in \mathbb{Z}$, we can now define a solution $\tilde{y}_i$ by

$$\tilde{y}_i = \mathcal{M}_{\gamma_{0,-\alpha_i-\pi/2}}(\tilde{\psi}_i) - \mathcal{M}_{\gamma_{0,-\alpha_i-3\pi/2}}(\tilde{\psi}_i) \tag{4.20}$$

It is easily verified that

$$\tilde{y}_i = \mathcal{M}_{\gamma_{a,-\alpha_i-\pi/2}}(\tilde{\psi}_i) - \mathcal{M}_{\gamma_{a,-\alpha_i-3\pi/2}}(\tilde{\psi}_i)$$

for any $a \in S(-\alpha_i - 3\pi/2, -\alpha_i - \pi/2)$, and hence it follows that $\tilde{y}_i$ is analytic in a left half-plane. Moreover, as will be shown presently, it has particular asymptotic properties.

In general, however, $\tilde{\psi}_i$ may grow too fast on the half lines $\gamma_{0,-\alpha_i-\pi/2}$ and $\gamma_{0,-\alpha_i-3\pi/2}$, and the Mellin transforms in (4.20) may not exist. In that case, we will replace these Mellin transforms by the sums of the Mellin transforms of the terms on the right-hand sides of (4.6) and (4.11), i.e., by the right-hand sides of (4.18) and (4.19), respectively. Thus we get the following extension of the definition given in (4.20):

$$\tilde{y}_i := y_i^- - M_i(M_i^0)^{-1} y_{i+N}^- - \sum_{j>i:\,\alpha_j<\alpha_{i+N}} C_i^j y_j^- \\ - \sum_{j<i:\,\alpha_j=\alpha_i} C_i^j(M_j^0)^{-1} y_{j+N}^- - C_{i0}M^{-1}y^{0\alpha_{i+N}^-}. \tag{4.21}$$

With the aid of (4.14), (4.15), and (4.16), we find

$$\tilde{y}_i = (I - e^{-2\pi i x}M_i)\, y_i^- - \sum_{j>i:\,\alpha_j<\alpha_{i+N}} C_i^j y_j^- - e^{-2\pi i x} \sum_{j<i:\,\alpha_j=\alpha_i} C_i^j y_j^- \\ - e^{-2\pi i x}C_{i0}(I - e^{-2\pi i x}M)^{-1}(M-I) \sum_{j:\,\alpha_i\leqslant\alpha_j<\alpha_{i+N}} C^{0j}y_j^-.$$

Noting that $C_i^j = C_{i0}C^{0j}$, using (3.9), and rearranging the terms of the above identity, we obtain

$$\tilde{y}_i = (1 - e^{-2\pi i x})\Big[\{M_i - C_{i0}(I - e^{-2\pi i x}M)^{-1}C^{0i}\}\, y_i^- \\ + \sum_{j<i:\,\alpha_j=\alpha_i} \{C_i^j - C_{i0}(I - e^{-2\pi i x}M)^{-1}C^{0j}\}\, y_j^- \\ - \sum_{j>i:\,\alpha_j<\alpha_{i+N}} C_{i0}(I - e^{-2\pi i x}M)^{-1}C^{0j}y_j^-\Big]. \tag{4.22}$$

Hence we conclude, with Theorem 3.6 and Proposition 4.3, that

$$\tilde{y}_i(x) = (1 - e^{-2\pi i x})\,\Gamma(x)\,\tilde{z}_i(x). \tag{4.23}$$

The following result is easily derived from (4.23) and Theorem 3.3.

PROPOSITION 4.5. *The functions* $\{\tilde{y}_{ip}: p\in\{1,...,m_i\},\ i\in\{1,...,N\}\}$ *form a fundamental system of solutions of the difference equation* (Δ_1), *analytic in a left half-plane. Moreover, there exists a positive number* δ *such that, for all* $i\in\mathbb{Z}$, $x^{-x}(e\lambda_i)^x\,\tilde{y}_i(x)$ *has subexponential growth as* $x\to\infty$ *in* $S[-3\pi/2, -\pi/2+\delta]$.

Concluding Remarks

We have derived the properties of the fundamental systems $Y^{\pm}$ and $\tilde{Y}$ from those of $Z^{\pm}$ and $\tilde{Z}$ by means of the relations given in Proposition 4.3 and formula (4.23). They can also be deduced directly from the properties of the solutions $\psi_i^{\pm}$ and $\tilde{\psi}_i$ of the differential equation (D_1). We shall briefly indicate how this can be done and how some of the results of Section 4 can be extended to a larger class of equations.

Let (D) be a differential equation of the type

$$\sum_{h=0}^{M} \sum_{l=0}^{m} a_{hl} t^h \left(t \frac{d}{dt} \right)^l \psi = 0. \tag{D}$$

Let $\{\psi_i\}_{i \in I}$ be a finite collection of solutions of (D) and $\{\gamma_i\}_{i \in I}$ a collection of half lines from O to ∞ such that ψ_i is analytic on γ_i and admits asymptotic expansions of the type (1.1) both as $t \to 0$ on γ_i and as $t \to \infty$ on γ_i. In view of Proposition 1.3, the Mellin transform $\mathscr{M}_{\gamma_i}(\psi_i)$ is analytic in a right half-plane whenever all but a finite number of coefficients of the asymptotic expansion of ψ_i at ∞ vanish. Furthermore, the sum $\sum_{i \in I} \mathscr{M}_{\gamma_i}(\psi_i)$ is analytic in a right half-plane if and only if the sum of the asymptotic expansions at ∞ of the functions ψ_i, $i \in I$, has but a finite number of nonzero coefficients. Similarly, this sum of Mellin transforms defines a function that is analytic in a left half-plane if and only if the sum of the asymptotic expansions at O has at most a finite number of nonzero coefficients.

If (D) has a regular singularity at the origin and an irregular one at ∞, the Mellin transform $\mathscr{M}_\gamma(\psi)$ of a solution ψ of (D) which decreases exponentially as $t \to \infty$ on the half line γ defines a solution of the corresponding difference equation $(\Delta) := (\mathscr{M}(D))$, analytic in a right half-plane. When (D) has no other singularities, it is possible to construct a fundamental system of solutions of (Δ) in this way. (Cf. the method proposed by Ramis, mentioned in the Introduction.) However, as there are no exponentially decreasing solutions of (D) at the origin, the construction of solutions of (Δ) that are analytic in a left half-plane is more complicated. The asymptotic expansions at the origin being convergent representations of the solutions of (D) in this case, one has to look for solutions of the differential equation with at most moderate growth at ∞ in some direction, whose sum either vanishes or is represented by a series of the form (1.1) with a finite number of nonzero coefficients. In the case considered in Section 4, the sum of the solutions of (D_1) used in the definition of $\tilde{y}_i$ is identically zero, as can be seen from (4.6), (4.11), and (4.21). Furthermore, it is clear from (4.21), (4.13), and Proposition 1.3, that, apart from at most a finite number, the singularities of $\tilde{y}_i$ in the right half-plane $\operatorname{Re} x > 0$ are determined by the asymptotic behaviour at ∞ of $\psi^{0\alpha_i^- + N}$.

A second important property of the fundamental systems $Y^{\pm}$ and $\tilde{Y}$ is their asymptotic behaviour at ∞. This can be deduced from the asymptotic properties of the solutions $\psi_i^{\pm}$ and $\tilde{\psi}_i$ of the differential equation (D_1) by means of the saddle point method. It is valid for all values of $\arg x$ for which the saddle point of the function $\exp(-\lambda_i t)\, t^{x-1}$ belongs to the sector where $\psi_i^{\pm}(t)\exp(\lambda_i t)$ or $\tilde{\psi}_i(t)\exp(\lambda_i t)$ has subexponential growth. The proof of this fact is very straightforward in the case of Y^+ and Y^- and a little more delicate in the case of $\tilde{Y}$. It is considerably more involved, as will be shown in Part II of this paper, when the singularity of the differential equation at ∞ is of a more complicated type than the one considered in Section 4.

If, at ∞, for all $p \in \{1, ..., m_i\}$, $\psi_{ip}^{\pm}$ $(\tilde{\psi}_{ip})$ is represented asymptotically by the formal solution $\hat{\psi}_{ip}$ given by (4.1), then it can be shown by means of the saddle point method, that $y_{ip}^{\pm}$ $(\tilde{y}_{ip})$ admits an asymptotic representation $\hat{y}_{ip}$ of the form

$$\hat{y}_{ip}(x) = x^x (e\lambda_i)^{-x} \exp(\tilde{q}_{ip}(x))\, x^{-\rho_{ip}-3/2} \tilde{g}_{ip}(x),$$

where $\tilde{q}_{ip}$ is a polynomial in $x^{1/k}$ of degree less than k and $\tilde{g}_{ip} \in \mathbb{C}[\![x^{-1/k}]\!][\log x]$.

Finally, we note that the connection formula (4.22) which relates the "left" fundamental system $\tilde{Y}$ of (Δ_1) to the "right" fundamental system Y^- is made up of two distinct ingredients: (i) the Stokes multipliers of level 1 at ∞ of the differential equation (D_1), here expressed in terms of the central connection matrices C_{i0} and C^{0j} of (D); and (ii) the monodromy matrix at the origin of (D), which is related to the asymptotic behaviour at ∞ of the solutions $\psi^{0\alpha}$ of (D_1) (its eigenvalues are the numbers $e^{2\pi i \rho_p^0}$, $p = 1, ..., m$, occurring in the expansion $\hat{\psi}_p^0$, cf. (4.1)) and which determines the location of the poles of $\tilde{Y}$ "to the right."

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REFERENCES

1. W. Balser, W. B. Jurkat, and D. A. Lutz, Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations, *J. Math. Anal. Appl.* **71** (1979), 48–94.
2. W. Balser, W. B. Jurkat, and D. A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities, I, *SIAM J. Math. Anal.* **12** (1981), 691–721.

3. W. Balser, W. B. Jurkat, and D. A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities, II, *SIAM J. Math. Anal.* **19** (1988), 398–443.
4. G. Doetsch, "Handbuch der Laplace Transformation, Band I, II," Birkhäuser-Verlag, Basel, 1950, 1955.
5. A. Duval, Equations aux différences algébriques: solutions méromorphes dans $\mathbb{C} - \mathbb{R}^-$. Système fondamental de solutions holomorphes dans un demi-plan, *in* "Equations différentielles et systèmes de Pfaff dans le champ complexe—II," Lecture Notes in Mathematics, Vol. 1015, pp. 102–135, Springer-Verlag, New York/Berlin, 1983.
6. A. Duval, Solutions irrégulières d'équations aux différences polynomiales, *in* "Equations différentielles et systèmes de Pfaff dans le champ complexe—II," Lecture Notes in Mathematics, Vol. 1015, pp. 64–101, Springer-Verlag, New York/Berlin, 1983.
7. J. Ecalle, "Les Fonctions Résurgentes, tome III," Publ. Math. d'Orsay, Université de Paris-Sud, Paris, 1985.
8. H. Galbrun, Sur la représentation des solutions d'une équation linéaire aux différences finies pour les grandes valuers de la variable, *Acta Math.* **36** (1913), 1–68.
9. H. Galbrun, Sur certaines solutions exceptionnelles d'une équation linéaire aux différences finies, *Bull. Soc. Math. France* **49** (1921), 206–241.
10. G. K. Immink, Asymptotic expansions with error bounds for solutions of difference equations of "level 1^+," *in* "Équations différentielles dans le champ complexe, Vol. I: Équations différentielles ordinaires," pp. 35–60, Publ. I.R.M.A., Strasbourg, 1988.
11. G. K. Immink, Resurgent functions and connection matrices for a linear homogeneous system of difference equations, *Funkcial. Ekvac.* **31** (1988), 197–219.
12. G. K. Immink, Note on the relation between Stokes multipliers and formal solutions of analytic differential equations, *SIAM J. Math. Anal.* **21** (1990), 782–792.
13. M. Kohno, A two point connection problem, *Hiroshima Math. J.* **9** (1979), 61–135.
14. B. Malgrange, Introduction aux travaux de J. Ecalle, *Enseign. Math.* **31** (1985), 261–282.
15. B. Malgrange, Equations différentielles linéaires et transformation de Fourier: Une introduction, *Ensaios Mat.* **1** (1989).
16. B. Malgrange, "Equations Différentielles à Coefficients Polynomiaux," Progress in Mathematics, Birkhäuser, Basel, 1991.
17. B. Malgrange, Fourier transform and differential equations, *in* "Recent Developments in Quantum Mechanics" (A. Boutet de Monvel *et al.*, Eds.), pp. 33–48, Kluwer, Dordrecht, 1991.
18. N. E. Nörlund, "Vorlesungen über Differenzenrechnung," Chelsea, New York, 1954.
19. K. Okubo, A global representation of a fundamental set of solutions and a Stokes phenomenon for a system of ordinary linear differential equations, *J. Math. Soc. Japan* **15** (1963), 268–288.
20. S. Pincherle, Sur la génération des systèmes récurrents au moyen d'une équation linéaire différentielle, *Acta Math.* **16** (1892), 341–363.
21. J. P. Ramis, Etude des solutions méromorphes des équations aux différences linéaires algébriques, manuscript.